Consolidation with Hysteresis in Sedimentary Basins

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Abstract

The necessitation for modeling various mechanical processes in porous media, such as consolidation and compaction, arises in several applied sciences. Within the context of sedimentary basins, we develop a model for consolidation with a permanent effect, a phenomena commonly observed in basin floors. In conjunction with the fundamental stress-porosity equation derived by Audet and Fowler, we utilize a multi-valued constraint relation to obtain a non-trivial model depicting the stress-porosity relationship of porous media during cyclical processes of loading/unloading. Irreversible changes to the porous structure of the medium relate to the inclusion of a permanent damage component, which leads to a lagging response, known as hysteresis, between the porosity and effective stress. The full rheology consists of elastic and viscoelastic components in addition to the previously mentioned damage component. However, the numerical simulations present in Section 3 focus on the viscoelastic and damaging behaviors exhibited by the classic Kelvin Voigt equation and our damage model respectively. Two numerical methods are implemented, where the first employs a well-known Runga-Kutta based solver ODE45, and the second, follows the work of Peszynska and Showalter, approximating the differential equations by an implicit backward difference scheme, where the resolvent operator is then recursively applied. We conclude the approach using the resolvent operator is stable and significantly more effective in comparison to ODE45.

Contents

1 Introduction

Consolidation is a well-known physical process in which the solid portion of a porous medium sustains a gradual decrease in its porosity due to the expulsion of pore fluid. Here, porous media consist namely of two components, a solid granular matrix, and interconnected voids, known as pores, through which fluid can freely travel. In our proposed setting of sedimentary basins, pore fluid is discharged as pressure is exerted on the medium's surface, where the pressure, given by force per unit area, is simultaneously distributed to both the exposed openings of the medium's surface pores as well as its solid matrix structure.

In this instance, the high volumetric stiffness of fluid, (usually groundwater), to the solid matrix means that the water initially absorbs any changes in pressure without altering the volume of the solid medium. This in turn creates excess pressure within the fluid, and due to seepage, water diffuses away from the pore spaces experiencing higher pressures [25]. This excess pore pressure serves as the central catalyst behind the initial evacuation of fluid from the medium, and as fluid exits the pore space, an increase in pressure transfers onto the granular structure, eventually culminating in a structural collapse, (i.e., a more efficient packing of the solid grain-like particles).

As these granules squeeze more tightly together, a decrease in the porosity, or volume fraction of space available to pore fluid within the medium, is readily observed. From this, we conclude that the theoretical framework of consolidation is closely related to the classical diffusion equation, the study of hydraulic conductivity, and the idea of effective stress pioneered by Karl Terzaghi. (See [10], [24], and [25] for details on the conceptual/historical development of effective stress). It is important to note that in fully saturated mediums, the collective effective stress of the solid in combination with the pressure of the pore fluid gives the total normal stress. Thus, effective stress is defined to be exactly this difference between the total stress and the neutral stress, (i.e., pore water pressure) [25]. Moreover, effective stress is the force responsible for the measurable effects with which we are concerned in this paper, namely consolidation and compaction.

Intergranular pressure, also known as effective pressure, is the force transferred between grains, (via surface-to-surface contact), due to the overburdening pressure from the weight of overlying material [17]. When this overburden pressure exceeds the pore fluid pressure, fluid ejects from the pore space, and the consolidation process begins to take effect. In essence, increasing effective pressure on the solid component of the medium is propagated by the expulsion of pore fluid while effectively reducing the medium's respective pore space, and this relationship between the effective pressure and fluid pressure leads to a fully coupled system describing the mechanics of the solid matrix and the fluid flow within it.

We further note that delayed reactions of the porosity to changes in effective stress indicate a phenomenon of interest, known as hysteresis, takes place during this process. Hysteretic, or lagging, behaviors are prevalent in differential models typically consisting of monotone evolution equations in conjunction with multi-valued constraint graphs [19]. Our research specifically follows references [5] and [12] leading to a pseudo-parabolic equation for porosity, where irreversible changes to the porous medium from our incorporated constraint graph result in the delayed stress-porosity response described.

In the context of sedimentology, (which is the study of the transportation, deposition, and diagenesis of sediments), gradual decreases in the porosity of porous media are often attributed to naturally occurring forces acting on the medium. For example, the accumulation of sediment on the upper floor of sedimentary basins produces changes in the underlying granular structures over time. In this case, the ejection of pore fluid directly results from the weight accrued via newly deposited material atop the medium. Various deformations to the medium's structure present themselves as the solid grains pack together more efficiently, and in many situations, these changes to the solid matrix are irreversible, which parallels conceptually to "permanent structural damage" taking place within the medium. The rheological behaviors of conventional models are widely accepted to be both irreversible and hysteretic in nature, and experimental results have been confirmed to exhibit these properties at depths of three to four kilometers in sedimentary basins [6].

In further elaboration, permanent damage frequently takes the form of rearrangement within granular particle structures or even as chemical reactions between them. In sedimentary basins, when the solid structure succumbs to the overburdening pressure from the aforementioned increase in weight of the overlying material, reductions in porosity occur as the medium consolidates. This force in conjunction with the hydrostatic pressure from trapped pore-fluid, (also known as lithostatic pressure, confining pressure, or vertical stress), causes the lithification of the granular structure at deeper levels of sediment. Interestingly enough this overpressuring occurs quite rapidly at depths deeper than 3 kilometers [5]. At these depths the stress of grain-to-grain contact becomes great enough to cause chemical dissolution, in which case minerals from the sediment diffuse into the pore-fluid, recrystallize, and eventually form sedimentary rocks such as sandstone, or siltstone [5] and [7]. Immense pressures sustained at these lower layers of the medium are characteristic of a highly related, yet more general process known as compaction outlined in greater detail in the next subsection.

Within the upper crust of sedimentary basins, (usually less than one kilometer deep), the behavior of consolidation in these sedimentary layers can be described as elastic due to the ability for particle grains to somewhat-reposition or expand when a load is removed or decreased [5], [12], and [23]. Situations like this can occur when storms blow vast amounts of sediment around basin floors, or through artificial dredging operations. These elastic or viscoelastic properties are best exemplified by recurring processes of pressurized loading and unloading in which the compressible structures regain some portion of the initial porosity from their pre-loaded state. In more realistic settings, as the stress input changes over time, an entire host of material factors influence the elasticity of the medium as well as the viscosity of the fluid saturating it.

Since sedimentary basins are usually relatively shallow, (on the order of kilometers), with respect to their width, (extending to tens or hundreds of kilometers), the rheology presented in Section 2 is well-approximated by one-dimensional depth-dependent models [1]. The major simplifying assumption we utilize is the fact that we are able to ignore the shear stress between grains since the ratio of shear to bulk modulus of the granular porous medium is essentially negligible [5]. This allows us to include remarkably general porosity-pressure constitutive relations which follow a normal consolidation curve, (see Figure 1), and the accumulated irreversible damage together with minor elastic or viscoelastic variations below this curve is represented by retraceable damage curves or rate-dependent loops [6] and [21].

In Section 2, we recall the basic two-phase solid-fluid model of fluid flow within a fully saturated and deformable porous medium, and we couple this base model with a strain-additive system in order to describe the combined viscoelastic hysteresis relation that occurs between the porosity and stress of a porous medium during the consolidation process. Numerical simulations demonstrating viscoelastic and hysteresis effects are presented in Section 3, where two types of consolidation are modeled. The first figure of each pair, (Left), is an example depicting the Kelvin Voigt model for viscoelasticity, in which loss of porosity in the medium is recoverable when the load or stress is released, however this recovery reacts to varying inputs in stress with an exponential time delay. In the second figure of each pair, (Right), our constructed damage constraint, $c(\cdot)$, is employed in an equation of consolidation behavior similar to that of Kelvin Voigt, but with the additional permanent damage component. Whenever a new minimum porosity is reached in time, if the load is released, the porosity remains unchanged until a stress of greater negative value is induced on the porous structure, creating the hysteretic (lagging) presence within our model's stress-porosity relationship.

On the last few pages, the references found include sources pertinent to the history/life of the "father of modern soil mechanics", Karl Terzaghi, detailing the trials and tribulations leading up to his unprecedented development of effective stress and the theory of consolidation he later became so widely known for. Some future objectives include providing information on sources imperative to this initial research, giving a brief history of Karl Terzaghi, his life accomplishments, as well as a short wrap up concerning where the frontier of consolidation modeling lies today, and commenting on how modern methods of computation have drastically impacted current thresholds of accuracy.

1.1 Relevance and Applications

Lagging behaviors between cause and effect characterize hysteresis, and they are found in a plentiful number of geophysical situations aside from consolidation modeling. Amazingly, this phenomenon takes place within the frameworks of electromagnetism, plasticity, phase transition, multiphase flow, adsorption-desorption processes, and even within applications as diverse as food processing and ecology [18]. Adsorption-desorption hysteresis is of some importance in modeling chemical interactions within reactor systems, carbon sequestration modeling, as well as wood science and engineering [20]. Hysteresis resulting from capillary pressures is also of considerable significance in endeavors regarding the simulation of multiphase flow dynamics. Undoubtedly hysteresis is fundamental aspect of the cyclic loading and unloading cycles endured by porous mediums undergoing consolidation as described in this paper and by other included sources such as [5], [7], and [12].

While consolidation falls underneath the umbrella of a more general process known as compaction, it does in fact have some subtle differences. Notably, in its most general connotation compaction may include the absence of pore fluid within the voids of the medium, whereas consolidation occurs strictly within saturated mediums, with no air trapped in the pore fluid, (i.e., the voids themselves are completely filled with liquid) [25]. Again, compaction is regularly associated with a more comprehensive class of processes, where in sedimentary basins, these occurrences may manifest at greater depths of the lithosphere, well beneath the sea-floor crust, when extreme sustained pressures are commonplace. Some references one might find useful pertaining to compaction that takes place in these particular settings are articles [5], [6], [7], and [12].

In this environment, the dissolution of minerals within the medium arises due to high contact-point pressures producing chemical/mineral solutions, which in turn causes the lithification of surrounding sedimentary structures [7]. On top of this, compaction includes abrupt mechanical procedures where pressure is applied, (in a rapid manner), to a granular medium such as soil, sand or gravel. In the context of civil engineering, this type of compaction is utilized in order to achieve the maximum dry density of a porous medium to ensure a solid structural foundation to build upon. This is accomplished through numerous types of heavy machinery exerting artificially created forces onto the soil-based foundations [9].

Consolidation and compaction modeling are both a major component of many coupled poromechanical systems, and our ability to model these scenarios is fundamental to a variety of the mentioned applications, including the mapping of subsurface groundwater flows as well as petroleum extraction. We turn our focus however to mainly the applications and relevance of consolidation modeling within this paper. In petroleum engineering, systems such as these play an important role in characterizing the pressurization that occurs during the hydraulic fracturing method utilized by oil drilling operations [12]. Hydraulic fracturing or "Fracking" as it is also known, is the process in which pressurized fluid is injected deep into the ground at an excavation site in order to reach oil deposits that would have been otherwise difficult to access.

Excavation sites are often located underwater in sedimentary basins such the Gulf of Mexico, where the Deep Water Horizon oil spill wrecked havoc on the otherwise pristine beaches of the U.S. Gulf Coast region in 2010. Information obtained from research concerning sedimentary basins, the geophysical properties that govern behaviors within them, and the potential hazards of exploiting their fossil fuel resources, is indispensable in the decision making process behind the future development of project sites. Because of this, some of the largest oil deposits, (i.e., the Destin Dome), have remained untouched. Environmental tragedies akin to the "B.P. oil spill", (as it is still regionally referred to as today), instill in us the importance of models such as the Kelvin Voigt model for viscoelasticity, that can be coupled with diffusive or convective processes to better exemplify the distribution of contaminants within subsurface flows [4] and [12]. Models of these type serve a critical role in predicting the environmental impact of extraction processes such as those previously mentioned, which has become an increasingly contentious topic across the globe today. Recent motions carried out by the current U.S. government administration to shut down a number of ongoing oil and gas infrastructure projects, (notably the Keystone XL Pipeline), and end/reject some major leasing permits to large oil conglomerates, (of particular significance, newly installed policies pertaining to the Arctic National Wildlife Refuge), have made this notion abundantly clear.

Several relevant applications of consolidation modeling, (and viscoelastic modeling), exist within the scope of geophysics and geological surveying when determining the origin of geological features. In modern glaciology, (the study of glaciers and permanent ice sheets), past consolidation cycles of sedimentary material can be used to ascertain the positions from which glaciers today have receded. While the massive weight of overlying ice decreases as the glacier melts away, the compressive force acting on the sediment underneath the glacier is removed, leaving the once underlying soil in an overconsolidated state that is identifiable by glaciologists.

Attributed to fact that ice sheets, ice caps, and glaciers are the largest source of fresh water on Earth, and they consist of gargantuan, slow-moving edifices of solid ice [4], these formations contribute significantly to watersheds imperative to societies across the world. Under their own immense weights, these structures sluggishly flow like colossal rivers over the landscape by solid-state creep processes such as the dislocation of crystal lattice structures [4]. In this case, creep is the tendency of a material to permanently deform from persistent subjection to mechanical stress. As ice sheets and glaciers calve/crevasse, stored water is unleashed, pouring through the resulting changes (cracks) in their structure. This fresh water eventually feeds from glaciers to the local watershed, and from ice sheets to the ocean via ice streams [4]. In one specific example, the draining action from ice sheet to ocean happens directly within several of the fjords that Greenland is renowned for. Discharged fresh water interacting with the salty ocean water, (which is inherently more dense), causes a pluming effect, and this has considerable implications when modeling the interface of the glacier, (part of the encompassing ice sheet), and the ocean. This interface plays a role in the ocean currents throughout these deep yet narrow inlets, and in the broader sense, influences the ocean circulation dominating the Northern Atlantic [13].

Throughout the entire planet, there are two major ice sheets, (in Greenland and Antarctica), a small number of ice caps, and numerous alpine glaciers in mountain ranges like those in Alaska, as well as the Himalayas, Karakorum, Alps, and Andes. Over time these megalithic ice structures have undoubtedly shaped the geomorphological history of Earth's surface [4]. Therefore, when appropriately applied, the study of viscoelastic behaviors and irreversible structural change allows scientists to effectively gain a greater understanding of the geological origination of our planet. Since past geological events often shape future occurrences, the ability to make more accurate predictions allows humans to better adapt/respond to threats in their changing environments. Clearly these types of models have the capability to benefit humanity. Either by mitigating some of the risks associated with environmental factors like volcanic eruptions, glacial melt, and watershed depletion, or by simply helping us gain a more in-depth understanding of how today's geological landscape came to be, and how it will continue to transform.

Finally, consolidation modeling has essential applications to civil/structural engineering projects such as bridges, dams, and large-scale building foundations. When appropriately incorporated into building/construction plans, these models can reduce the risk of catastrophic failure due to underlying issues with foundational support. The concepts of seepage and settlement are often discussed under these notions. None other than Karl Terzaghi, through careful inquiry, introduced an efficient method of preventing dam bursts that had previously taken place due to a seepage related phenomenon called piping. His approach was to provide a heavy filter, consisting of a blanketed mass of pervious soil on which sat the weight of additional backfill, at the critical failure location, (usually downstream near the "toe" of the dam) [10]. The theory of consolidation and effective stress was crucial to both the realization and facilitation of this ingenious solution. Practical uses may even be found in biology/biomechanics where consolidation theory has helped to better describe internally induced stresses on the brain, (those caused by cancerous or non-cancerous tumors), or to describe brain deformations from subjection to mechanical stresses during surgical operations [8] and [16].

2 Modeling Consolidation

In the setting of sedimentary basins, consolidation occurs within fully saturated porous media when under the pressure of its own weight or newly accumulated sediment, subsurface layers experience a gradual reduction in pore space due to the evacuation of pore water from within the granular structure. As pore fluid is displaced, the grains making up the solid medium endure higher levels of pressure at their contact points. This causes the sedimentary particles to rearrange, or pack, together in a more efficient manner. Clearly, there are physical situations which warrant this rearrangement be irreversible, thus necessitating the modeling for such behavior. The development and implementation of this "damaging" property is accomplished through the tactful construction and use of a set-valued constraint graph, $c(\cdot)$.

Contrarily, the effects of loading and unloading on the medium in the uppermost layers of these basins can be characterized as either elastic or viscoelastic, where changes in applied pressure result in reversible trends to the overall porosity of the solid structure. In idealistic, or fully elastic scenarios, when a load is removed the structure immediately regains the entirety of the initial porosity from its previous, pre-loaded, or pre-consolidated, state. In the viscoelastic case, the viscosity of the fluid medium effects the rate at which the solid portion regains its initial porosity, and this cycle of loading and unloading the medium follows a normal consolidation curve, as later shown in Figure 1.

The system (10) describes these required relationships as three equations, numerically simulated in Section 3. The classically known Kelvin Voigt model is used to model the stress-porosity response of viscoelastic consolidation, and a model incorporating a permanent damage component is expressed similarly to the Kelvin Voigt model. These permanent, time-dependent reductions in pore space require the porosity-stress equation be equipped with our multi-valued constraint graph, (i.e., the set-valued elbow graph), $c(\cdot)$. In the rheology, these models are given by (10b) and (10c) respectively.

2.1 The Porosity-Stress Equation

Let us establish the necessary mathematical notation as we derive our model from some otherwise known conservation equations. Following references [5] and [12], our basic differential model is obtained by utilizing the three equations: conservation of fluid, conservation of solid, and Darcy's Law. We first begin by considering a standard model for a two-phase fully-saturated sediment in which *porosity* $\phi(x,t)$ is the volume fraction of the medium available to fluid, *fluid velocity* is given by $\mathbf{v}_f(x,t)$ and *solid velocity* is denoted $\mathbf{v}_s(x,t)$; it is assumed that the fluid and solid densities ρ_f, ρ_s are constant. Here it is assumed that $\rho_s > \rho_f$ due to the fact that if this were not the case, buoyancy of the solid granules would occur within the fluid medium. *Pressure* of the fluid is given as $p(x,t)$, where μ is the viscosity, and $\kappa(\phi)$ is the permeability of the solid matrix. The consolidation model is given by the system:

$$\frac{\partial \rho_f \phi}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_f \phi \mathbf{v}_f) = \rho_f F, \tag{1a}$$

$$\frac{\partial \rho_s (1-\phi)}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_s (1-\phi) \mathbf{v}_s) = 0\,, \tag{1b}$$

$$\phi(\mathbf{v}_f - \mathbf{v}_s) = -\frac{\kappa(\phi)}{\mu} \boldsymbol{\nabla} p. \tag{1c}$$

The conservation of fluid is given by the equation (1a), and the conservation of solid is similarly given by the equation (1b). Darcy's Law, or the flow of a fluid through a porous medium, is represented by (1c). We initiate the derivation our model by utilizing (1a) and (1b) through rearranging and simplifying some terms. We assume that the densities are constant, so factoring out the conservation equations respective densities ρ_f and ρ_s, and dividing them out yields the pair of equations:

$$\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot (\phi \mathbf{v}_f) - F = 0, \qquad \frac{\partial (1-\phi)}{\partial t} + \boldsymbol{\nabla} \cdot ((1-\phi) \mathbf{v}_s) = 0.$$

Taking the sum of the two conservation equations above and further simplifying by distributing the partial derivative with respect to t to $(1-\phi)$, we cancel terms to obtain:

$$\boldsymbol{\nabla} \cdot (\phi \mathbf{v}_f + (1-\phi) \mathbf{v}_s) = F.$$

Now, letting $\mathbf{v}$ be defined by $\mathbf{v} \equiv \phi \mathbf{v}_f + (1-\phi)\mathbf{v}_s$ gives us:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = F.$$

The fluid and solid displacements will be primarily vertical, due to the width to depth ratio of the basin, so we shall assume that $\mathbf{v}$ is irrotational, that is, $\boldsymbol{\nabla} \times \mathbf{v} = 0$. We utilize Helmholtz Theorem, that implies that $\mathbf{v} = \boldsymbol{\nabla} f$, where f is a pseudo-potential function. This results in $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = F$, that is, $\Delta f = F$ in Ω. Assuming the normal flux $\mathbf{n} \cdot \mathbf{v}$ is known on the boundary $\partial\Omega$, we can then solve the Neumann Problem for f.

In summary when $\Delta f = F$ in Ω, and the normal composite flux is given by $\frac{\partial f}{\partial \mathbf{n}} = \mathbf{n} \cdot \mathbf{v}$, we denote our solution to the above equation to Neumann boundary conditions by $f = \Delta^{-1} F$, so that we have:

$$\phi \mathbf{v}_f + (1-\phi)\mathbf{v}_s \equiv \mathbf{v} = \boldsymbol{\nabla} f = \boldsymbol{\nabla} \Delta^{-1} F.$$

Therefore, by distributing $\mathbf{v}_s$ to $(1-\phi)$, we obtain $\phi \mathbf{v}_f + \mathbf{v}_s - \phi \mathbf{v}_s = \boldsymbol{\nabla} \Delta^{-1} F$, and then combining terms and solving for $\mathbf{v}_s$ gives us:

$$\mathbf{v}_s = -\phi(\mathbf{v}_f - \mathbf{v}_s) + \boldsymbol{\nabla} \Delta^{-1} F.$$

Additionally, we substitute $\frac{-\kappa(\phi)}{\mu} \boldsymbol{\nabla} p$ for the term $\phi(\mathbf{v}_f - \mathbf{v}_s)$ in the above equation via application of Darcy's Law (1c), and this produces:

$$\mathbf{v}_s = \frac{\kappa(\phi)}{\mu} \boldsymbol{\nabla} p + \boldsymbol{\nabla} \Delta^{-1} F.$$

From here, substituting the equivalent form of $\mathbf{v}_s$ into the conservation equation for solids (1b) and moving all terms from the right to the left side gives us the following *porosity-pressure equation*:

$$\frac{\partial \phi}{\partial t} - \boldsymbol{\nabla} \cdot \left((1-\phi) \left(\frac{\kappa(\phi)}{\mu} \boldsymbol{\nabla} p + \boldsymbol{\nabla} \Delta^{-1} F \right) \right) = 0\,. \tag{2}$$

The *effective pressure* of the solid matrix is notated as $p_{eff}(x,t)$. Since we shall ignore the shear stress in the matrix, the *effective stress* is given by the tensor $\sigma\delta = -p_{eff}(x,t)\delta$. Here δ is the identity tensor. The effective stress shares the load of the *overburden pressure* P with the fluid pressure, so we have:

$$P = p + p_{eff} = p - \sigma. \tag{3}$$

Finally from (2) and (3) we get the fundamental *porosity-stress equation*:

$$\frac{\partial \phi}{\partial t} - \boldsymbol{\nabla} \cdot \left((1-\phi) \left(\frac{\kappa(\phi)}{\mu} \boldsymbol{\nabla} \sigma + \frac{\kappa(\phi)}{\mu} \boldsymbol{\nabla} P + \boldsymbol{\nabla} \Delta^{-1} F \right) \right) = 0 \,. \tag{4}$$

We note the above equation must be supplemented with the *stress-porosity* constitutive relation with hysteretic behavior to be applicable within the context of irreversible damage.

2.2 The Rheology

The classical constitutive behaviour for a *viscoelastic* solid is described by the equation

$$\sigma = E(\phi) + \eta \tfrac{\partial \phi}{\partial t} \tag{5}$$

in which $E(\cdot)$ is a real-valued monotone function and $\eta > 0$. Known in the literature as the *Kelvin-Voigt model*, equation (5) consists of the stress-additive combination of an elastic component $E(\phi)$ and a viscous component $\eta \frac{\partial \phi}{\partial t}$. Variations in the stress yield a porosity following the elastic response $E(\phi)$ with an exponential time delay. The substitution of (5) into (2) leads to a *pseudo-parabolic* equation for porosity:

$$\frac{\partial \phi}{\partial t} - \boldsymbol{\nabla} \cdot \Big((1-\phi) \tfrac{\kappa(\phi)}{\mu} \boldsymbol{\nabla} (E(\phi) + \eta \tfrac{\partial \phi}{\partial t}) + (1-\phi) \tfrac{\kappa(\phi)}{\mu} \boldsymbol{\nabla} P \Big) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \Delta^{-1} F). \tag{6}$$

See references by Bohm and Showalter [3], and Holland and Showalter [12] for additional information with regards to the well-posedness of initial-boundary-value problems for such equations. We define the elliptic operator that appears in (6) by:

$$A_\phi = -\boldsymbol{\nabla} \cdot (1-\phi) \tfrac{\kappa(\phi)}{\mu} \boldsymbol{\nabla}.$$

With this notation, the equation (6) takes the form:

$$\frac{\partial \phi}{\partial t} + A_\phi E(\phi) + \eta A_\phi \frac{\partial \phi}{\partial t} = -A_\phi P + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \Delta^{-1} F). \tag{7}$$

Notice in the purely elastic case where $\eta = 0$, equation (6) is the parabolic *porous medium equation*. An adequate model for consolidating processes in porous media must include both the instantaneous elastic and rate-dependent viscous effects of variation of the solid strain (5) in addition to the permanent reduction of porosity with increasing effective pressure. This is described by the *normal consolidation curve*, $E(\phi) = \sigma = -p_{eff}$ and usually specified by the inverse function $\phi = K(\sigma)$.

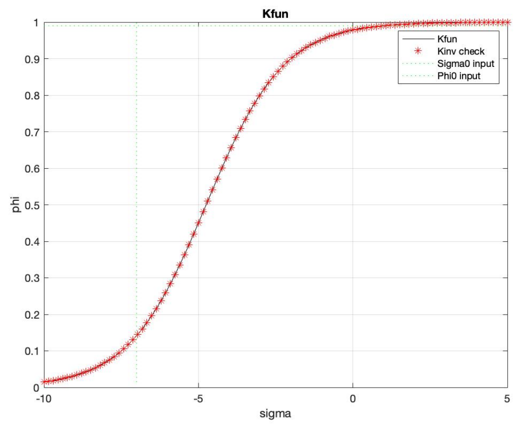


Figure 1: This *normal consolidation curve* depicts the relationship between the applied stress σ and the porosity, (or void-ratio ϕ), given by: $\phi = K_3(\sigma)$. As stress decreases in value, the portion of void-space available to fluid within the medium decreases in response.

This function describes the change in porosity as the effective stress of the porous matrix decreases due to an increase in load. We shall write this relationship in terms of its inverse, $\phi = K(\sigma)$. It models the effects of non-reversible *consolidation* of the porous structure due to increasing effective pressure. For these changes to be irreversible as pressure is relieved, this function can not be retraced, leading to *hysteresis* within the model. We shall begin to formulate this central component of the full model by defining the maximal monotone *constraint relation*, i.e., set-valued function:

$$c(s) = \begin{cases} \{0\} \text{ if } s < 0, \\ [0, +\infty) \text{ if } s = 0. \end{cases} \tag{8}$$

The relation $y \in c(x)$ is equivalent to $y \geq 0,\ x \leq 0$ and $xy = 0$. The hysteresis component of the consolidation due to compaction of the porous structure will be given by the ordinary differential equation of the form:

$$\frac{d\phi(t)}{dt} + c\left(\phi(t) - K(\sigma(t))\right) \ni 0. \tag{9}$$

It gives the porosity resulting from a given stress history. As stress $\sigma(t)$ decreases from 0, the porosity stays constant until it reaches the graph of $K(\cdot)$, and then it decreases along that graph. If $\sigma(t)$ changes direction at some value σ_0 and increases, the porosity stays constant until the stress decreases back to σ_0, and then the porosity resumes its decrease along $K(\cdot)$. This *hysteresis relation* $\phi = \mathbf{H}(\sigma)$ defined by (9) supplements the stress-porosity constitutive equation (5) to describe the *damage* component of the rheology in the form of permanent consolidation within the medium.

The full model is a combination of these three components acting along the normal consolidation curve: an instantaneous elastic component ϕ_1, an exponentially-delayed visco-elastic component ϕ_2, and the irreversible damage component ϕ_3.

$$\sigma = E_1(\phi_1), \tag{10a}$$

$$\tfrac{\partial \phi_2}{\partial t} + \tfrac{1}{\eta}\left(E_2(\phi_2) - \sigma\right) = 0, \tag{10b}$$

$$\tfrac{\partial \phi_3}{\partial t} + c(\phi_3 - K_3(\sigma)) \ni 0, \tag{10c}$$

$$\tfrac{\partial \phi_1}{\partial t} + \mathcal{A}_\phi \sigma - \tfrac{1}{\eta}\left(E_2(\phi_2) - \sigma\right) - c(\phi_3 - K_3(\sigma)) = -A_\phi P + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \Delta^{-1} F). \tag{10d}$$

Here, the first equation represents the non-linear elastic part of the model, the second is the visco-elastic portion, and the third is the non-recoverable "damage" to the granular structure. Once again, damage meaning once a load exerts some force crossing the threshold of a specific overburdening pressure to the solid matrix, the solid grains begin to rearrange, and at this point they can never regain their original position/placement. The *total porosity* consists of a *strain-additive* combination of these components into the single term given by: $\phi = \phi_1 + \phi_2 + \phi_3$, and this replacement in (10d) yields the equivalent equation:

$$\tfrac{\partial \phi}{\partial t} + \mathcal{A}_\phi \sigma = -A_\phi P + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \Delta^{-1} F).$$

To specify an appropriate initial-boundary-value problem for the system (10), we need to specify Dirichlet boundary conditions on the effective stress σ and the initial condition on each of the porosity components, $\phi_1(x, 0)$, $\phi_2(x, 0)$, $\phi_3(x, 0)$.

3 Rheology Simulation

Although the task of simulation is to numerically solve for an approximation of the solution of the full system (10), we shall focus on how to compute the total porosity ϕ from a given effective stress σ. The first component ϕ_1 expressed in (10a) is obtained from a standard (nonlinear) solver to invert E_1. Standard ODE solvers are adequate for the visco-elastic component ϕ_2 obtained from (10b). We recognize the endeavor of simulating the full strain-additive model as worthy of future investigation. The development of literature surrounding simplified model simulations would also have value in terms of providing educational assistance appertaining to introductory-reader-level conceptualization of these fundamental processes.

In simulating (10c), the model of the permanent damage component ϕ_3, we shall compare two numerical approaches. The first consists of approximating the singular constraint function $c(\cdot)$ by a Lipschitz continuous function (depending on a regularizing parameter $\varepsilon > 0$) and then applying a standard MATLAB code to approximate the solution of the corresponding initial-value problem. The second and more effective approach developed in Peszynska and Showalter [18] is to approximate the differential equation by its *implicit backward-difference* equation with time-step $\tau > 0$. This involves the recursive use of the resolvent operator $(I + \tau c)^{-1}$. Moreover, it is a contraction, so stability is immediate, and because of a particular property of the constraint function, namely, $\tau c(\cdot) = c(\cdot)$ for each $\tau > 0$, the solution of (10c) is *rate-independent*. We shall illustrate these two approaches for the initial-value problem,

$$\tfrac{d}{dt}\phi(t) + c\big(\phi(t) - K(\sigma(t))\big) = 0, \tag{11a}$$

$$\phi(0) = \phi_0, \tag{11b}$$

for which $K(\sigma)$ is the monotone normal consolidation curve, ϕ_0 is the initial porosity of the porous medium's granular structure, and $c(\cdot)$ is the multi-valued constraint graph (8).

Regularized Formulation

In order to solve (11) numerically using an explicit RK45 method (ode45) code, we must write it in the form $u'(t) = f(t,u)$, where $f(t,u)$ is specifically a single-valued function. We opt to use the approximation of the constraint function given by $c_\epsilon(u) = \frac{1}{\varepsilon}u_+$ where $u_+ = \max(u,0)$ is the positive part of u. This is the *Yosida approximation* of the maximal monotone graph c given in general by $c_\epsilon = \frac{1}{\varepsilon}(I - (I+\varepsilon c)^{-1})$. It is always a monotone Lipschitz continuous function. In summary, the first approach to approximating (11) is to solve the equation,

$$\tfrac{d}{dt}\phi(t) + \tfrac{1}{\varepsilon}\big(\phi(t) - K(\sigma(t))\big)_+ = 0, \tag{12}$$

with a standard ODE solver. Again, here we choose the well-developed ODE45 "blackbox" solver, (based on high order Runga-Kutta methods). Note that in the region where $\phi > K(\sigma)$ the equation (12) has the same form as the visco-elastic component (10b) with $K = E_2^{-1}$ and $\tau = \eta$, but where $\phi < K(\sigma)$ we have $\phi'(t) = 0$.

Since $K(\cdot)$ is monotone and the equation is linear in ϕ, the solution is given locally for $t_\ell \le t$ by:

$$\phi(t) = \begin{cases} e^{\frac{-t+t_\ell}{\varepsilon}}\phi_\ell + \int_{t_\ell}^{t} e^{\frac{-t+s}{\varepsilon}} K(\sigma(s))\,ds \text{ if } \phi(t) \ge K(\sigma(t)), \\ \phi_\ell \ \text{ if } \ \phi(t) < K(\sigma(t)), \end{cases}$$

where t_ℓ was the last previous time that $\phi_\ell = K(\sigma(t_\ell))$. Thus, this approximate solution is similar to that of (10b) with $\tau = \eta$ when $\sigma(t)$ is decreasing.

Implicit Formulation

At this stage, we follow [18] and set up the implicit difference scheme for equation (12) obtained by approximating the derivative with a backward difference quotient,

$$\frac{1}{\tau}(\phi^n - \phi^{n-1}) + c(\phi^n - K(\sigma(t_n))) = 0 \text{ for } n \ge 1,\ \phi^0 = \phi_0, \tag{13}$$

and we rewrite the equation in the form:

$$\phi^n - K(\sigma(t_n)) + \tau c(\phi^n - K(\sigma(t_n))) = \phi^{n-1} - K(\sigma(t_n)).$$

The left side is $(I + \tau c)(\phi^n - K(\sigma(t_n)))$, so the solution is given recursively by:

$$\phi^n = (I + \tau c)^{-1}(\phi^{n-1} - K(\sigma(t_n))) + K(\sigma(t_n)),\ n \geq 1,\ \phi^0 = \phi(0). \tag{14}$$

Note here that the resolvent is defined by the negative portion of the identity function, and zero otherwise:

$$(I + \tau c)^{-1}(x) = (I + c)^{-1}(x) = x_- = \begin{cases} x, & x < 0, \\ 0, & x \geq 0. \end{cases} \tag{15}$$

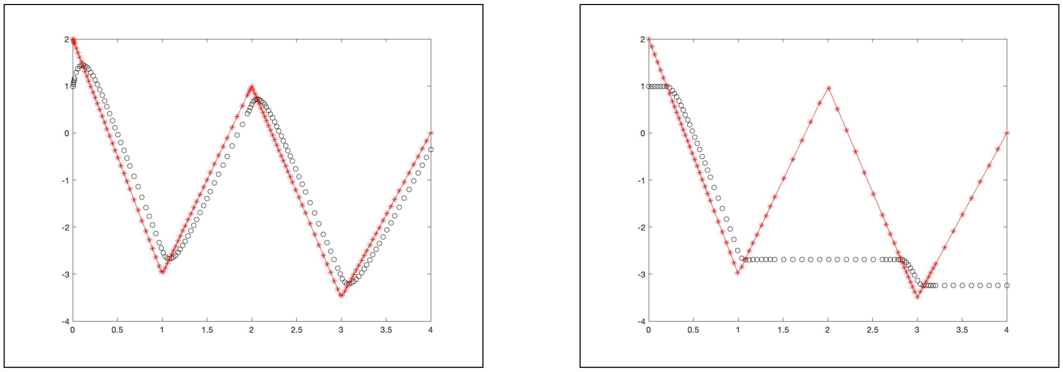
This tells us that the resolvent is the projection of $x \in \mathbb{R}$ onto the convex set $(-\infty, 0]$, and moreover, it shows that:

$$\phi^n = \begin{cases} \phi^{n-1}, & \phi^{n-1} \leq K(\sigma(t_n)), \\ K(\sigma(t_n)), & \phi^{n-1} \geq K(\sigma(t_n)). \end{cases} \tag{16}$$

Thus, we determine that (16) is the solution of the approximation given by equation (13).

3.1 Numerical Simulations

Figure 1 (shown in subsection 2.2) illustrates the non-linear elastic stress/porosity relationship as the normal consolidation curve given by the equation $\phi = K(\sigma)$. Here, the domain for our stress variable σ is scaled arbitrarily from [-10,5] and the the range for our dependent variable ϕ, which represents the void ratio of the porous medium, is the interval [0,1]. This is due to the fact that the porosity, or void ratio, is defined as the volume fraction of space available to pore fluid with respect to the total volume of the porous medium. In other words, the value for porosity must always be some fraction . The curve depicted in Figure 1 is used in four simulations, (demonstrated by figures 3 and 4), to lend more realistic physical consolidation behavior as shown by experimental data in sources such as [2], [24], and [25].

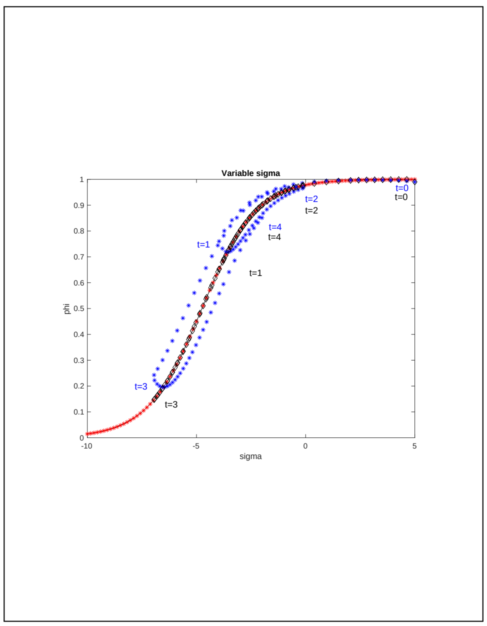


(a) (b)

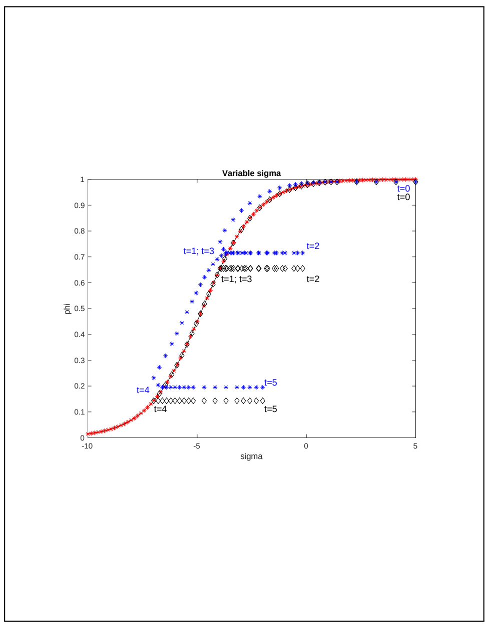
Figure 2: The viscoelastic stress-porosity response relationship is depicted, where in figure (a) there is no damage component, and the porosity (given by black circles) has an exponentially delayed response to changes in the stress (given by red stars). In figure (b), the same delayed response is observed, however, in this case the damage component is clearly present. (i.e. a new minimum porosity value remains constant after every response to a stress of a greater negative value until acted upon by a stress of higher intensity).

We first consider the (non-physical) special case of $K(\sigma) = \sigma$ in Figure 2 to illustrate the effect of the approximated constraint function c_ε while applying a piecewise linear stress input that takes successive extreme values: $2,\ -3,\ 1,\ -3.5,\ 0$. The graph of the described stress function is depicted by red stars. Whenever the approximated constraint c_ε is removed and replaced by $\frac{1}{\varepsilon}I$ in equation (12), where I is the identity function, the solution is given by the black circles in (a). However, when we use the original regularized formulation (12), the solution is shown by the black circles in (b). That is, the solution of porosity $\phi(t)$ is stress respondent, following $\sigma(t)$ downward with an exponential delay, but it remains constant when stress $\sigma(t)$ is increasing above the last σ producing a minimum ϕ.

Since we disregard shear stress in our specified setting, stress σ is equal to negative effective pressure $-p_{eff}$, where effective pressure is defined as force per unit area required to deform the solid. Positive pressure values occur when the enclosed spaces of the porous medium have higher pressure values than the space surrounding it. This in-turn forces the evacuation of fluid from the medium, (due to seepage), and ultimately causes a collapse within the solid portion of the medium, where in our context, sedimentary particles pack more closely together. Conversely, negative pressure values occur when the enclosed space of the solid matrix has lower pressure values than its surrounding environment, and this essentially forces the absorption of fluid into the void space of the porous medium. This elongates or pushes apart the granular structure depending on the specific media involved. Notice, the initial stress in these figures begins with a positive value of $\sigma = 2$ at time $t = 0$, where graph (a) of Figure 2 shows what could be interpreted as an expansion of the initial pore space, and (b) preserves the minimum achieved porosity value whenever the input causes $K(\sigma)$ to produce ϕ values above this minimum. The less accurate numerical solution given by "blue stars" is calculated utilizing the explicit ordinary differential equation solver ODE45, while the more accurate solution given by "black diamonds" is found using the implicit resolvent method.

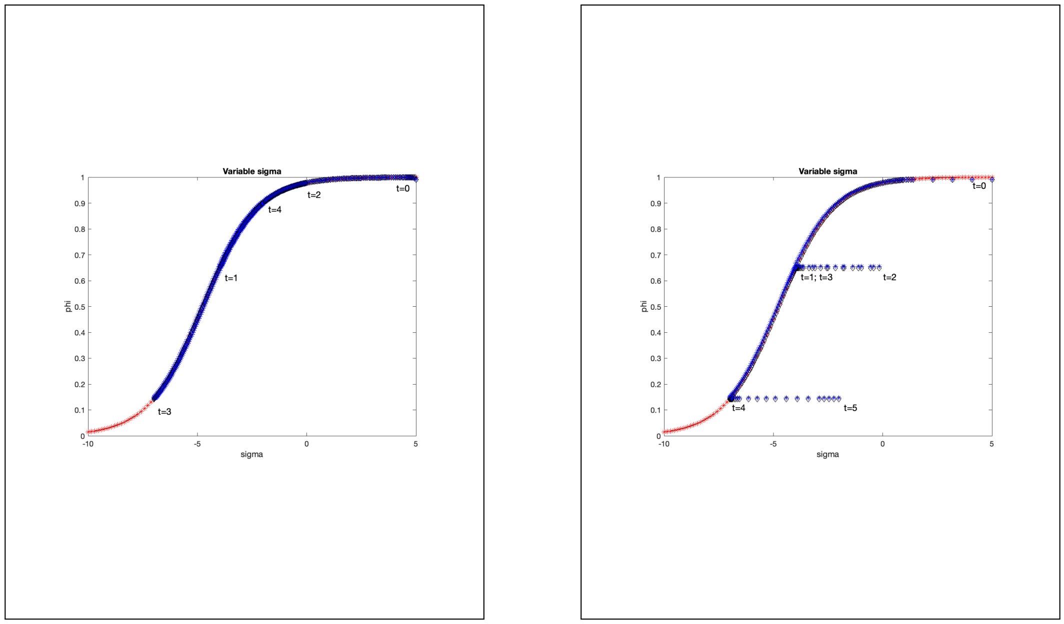


(a)



(b)

Figure 3: Two solutions are given for both purely viscoelastic behavior on the consolidation curve, (a), as well as viscoelastic consolidation with our proposed damage constraint, (b), when $\epsilon = 1$. While the ODE45 method, given by "blue stars", somewhat captures the desired behavior of both types of consolidation along the normal consolidation curve, it does less than a superior job in comparison to the resolvent method, given by "black diamonds".



(a) (b)

Figure 4: Purely viscoelastic behavior is portrayed on the consolidation curve in (a), and viscoelastic behavior with the permanent damage constraint on the consolidation curve is shown in (b), now with higher accuracy for the ODE45 method, (comparable to the resolvent solution), where in this case we have $\epsilon = .1$

Assessing the graphical outputs of these physical model simulations, (which occur along the normal consolidation curve in figure 1), we conclude that the implicit difference approximation scheme solved using the resolvent method [outlined by Dr. Malgorzata Peszynska and Dr. Ralph Showalter in their most recent publication, *"Approximation of Hysteresis Functionals"*] in (18) is the better choice. This is a result of the inherent stability of this resolvent operator, $(I + \tau c)^{-1}$, (as it is a contraction within the literature for operator theory in Hilbert spaces). In either case, the utilization of the above-mentioned method yields more efficient approximations to the initial value problem given by equation (11), in comparison to that of the explicit Runge-Kutta based ODE45. Depicting our desired damaging behavior, Figures 3 and 4 (b) show that when the porosity changes due to varied inputs in stress, as stress is relieved, (i.e. goes towards zero), the porosity remains constant until the moment the stress value once again reaches the consolidation curve, upon which it proceeds downward.

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